

ASYMPTOTIC SEQUENTIAL RADEMACHER COMPLEXITY OF A FINITE FUNCTION CLASS

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ABSTRACT. For a finite function class we describe the large sample limit of the sequential Rademacher complexity in terms of the viscosity solution of a G -heat equation. In the language of Peng's sublinear expectation theory, the same quantity equals to the expected value of the largest order statistics of a multidimensional G -normal random variable. We illustrate this result by deriving upper and lower bounds for the asymptotic sequential Rademacher complexity.

1. PRELIMINARIES

The notion of sequential Rademacher complexity was introduced in [10] (see also [11, 12]). Let $(\varepsilon_i)_{i=1}^n$ be independent Rademacher random variables: $\mathbf{P}(\varepsilon_i = 1) = \mathbf{P}(\varepsilon_i = -1) = 1/2$. Consider a set \mathcal{Z} , endowed with a σ -algebra \mathcal{G} , and a collection \mathcal{F} of Borel measurable functions $f : \mathcal{Z} \mapsto \mathbb{R}$. For any sequence of functions $z_n : \{-1, 1\}^{n-1} \mapsto \mathcal{Z}$, $n \geq 1$, where z_1 is simply an element of \mathcal{Z} , put

$$\mathfrak{R}_n(\mathcal{F}, z_1^n) = \frac{1}{\sqrt{n}} \mathbf{E} \sup_{f \in \mathcal{F}} \sum_{t=1}^n \varepsilon_t f(z_t(\varepsilon_1^{t-1})).$$

By a_1^n we denote a sequence (a_1, \dots, a_n) . The *sequential Rademacher complexity* of the function class \mathcal{F} is defined by

$$\mathfrak{R}_n(\mathcal{F}) = \sup_{z_1^n} \mathfrak{R}_n(\mathcal{F}, z_1^n). \quad (1.1)$$

The incentives to study this quantity come from the online learning theory, where on every round t a learner picks an element q_t from the set \mathcal{Q} of all probability distributions defined on the Borel σ -algebra of the metric space \mathcal{F} , and an adversary picks an element $z_t \in \mathcal{Z}$. The value $\int_{\mathcal{F}} f(z_t) q_t(df)$ determines the loss of the learner. The normalized cumulative regret over n rounds is defined by

$$\mathcal{R}_n(q_1^n, z_1^n) = \frac{1}{\sqrt{n}} \left(\sum_{t=1}^n \int_{\mathcal{F}} f(z_t) q_t(df) - \inf_{f \in \mathcal{F}} \sum_{t=1}^n f(z_t) \right).$$

This quantity compares the regret of the randomized strategy q_1^n with the regret of a best deterministic decision, taken in hindsight. Choosing their strategies, the learner

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and the adversary can use the information on all previous moves. Without going into the details, we only define the value of the repeated two-player game:

$$\mathcal{V}_n(\mathcal{F}) = \inf_{q_1 \in \mathcal{Q}} \sup_{z_1 \in \mathcal{Z}} \dots \inf_{q_n \in \mathcal{Q}} \sup_{z_n \in \mathcal{Z}} \mathcal{R}_n(q_1^n, z_1^n).$$

Typically, the sum $n^{-1/2} \sum_{t=1}^n \int_{\mathcal{F}} f(z_t) q_t(df)$ grows linearly in \sqrt{n} . The class \mathcal{F} is called *learnable* if:

$$\limsup_{n \rightarrow \infty} \frac{\mathcal{V}_n(\mathcal{F})}{\sqrt{n}} = 0.$$

The following nice estimate was obtained in [10, Theorem 2]: $\mathcal{V}_n(\mathcal{F}) \leq 2\mathfrak{R}_n(\mathcal{F})$. In the model of the supervised learning a similar lower bound also holds true: see [11, Proposition 9].

In the sequel we assume that the class \mathcal{F} is *finite*: $\mathcal{F} = \{f_1, \dots, f_m\}$, and its elements are uniformly bounded: $|f_i| \leq b$. Any such class is learnable: $\mathfrak{R}_n(\mathcal{F}) \leq b\sqrt{2 \ln m}$ (see [10, Lemma 5], [12, Lemma 1]). The goal of the present note is to characterize the quantity

$$\mathfrak{R}^a(\mathcal{F}) = \lim_{n \rightarrow \infty} \mathfrak{R}_n(\mathcal{F}),$$

which we call the *asymptotic sequential Rademacher complexity* of \mathcal{F} .

The mentioned estimate of $\mathfrak{R}_n(\mathcal{F})$ implies the inequality

$$\mathfrak{R}^a(\mathcal{F}) \leq b\sqrt{2 \ln m}. \quad (1.2)$$

Note, that (1.2) does not take into account the structure of the set \mathcal{F} . To get an insight into what features of \mathcal{F} are essential, let us consider the *Rademacher complexity*: a well established notion of a statistical learning theory, where the data z_t are assumed to be independent and identically distributed. Let $(Z_t)_{t=1}^n$ be a sequence of i.i.d. random variables with values in \mathcal{Z} . It is assumed also that $(Z_t)_{t=1}^n$ are independent from $(\varepsilon_t)_{t=1}^n$. The Rademacher complexity of a function class \mathcal{F} is defined by (see, e.g., [14, 18])

$$\mathfrak{R}_n^{iid}(\mathcal{F}) = \frac{1}{\sqrt{n}} \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{t=1}^n \varepsilon_t f(Z_t). \quad (1.3)$$

The role of this quantity in the statistical learning theory is similar to the role of (1.1) in the online learning theory.

For the case of a finite class $\mathcal{F} = \{f_1, \dots, f_m\}$ one may rewrite (1.3) as

$$\mathfrak{R}_n^{iid}(\mathcal{F}) = \mathbb{E} g \left(\sum_{t=1}^n \frac{\varepsilon_t F(Z_t)}{\sqrt{n}} \right), \quad g(x) = \max\{x_1, \dots, x_m\},$$

where $F(z) = (f_1(z), \dots, f_m(z))$. Although g is not bounded, the validity of the central limit theorem can be established with the use of the Hoeffding inequality as in [1, Lemma A.11] (see also the proof of Theorem 2.2 below). Let Σ be the covariance matrix of $\varepsilon F(Z)$, where (ε, Z) is distributed as (ε_t, Z_t) . Then

$$\mathfrak{R}^{a,iid}(\mathcal{F}) := \lim_{n \rightarrow \infty} \mathfrak{R}_n^{iid}(\mathcal{F}) = \mathbb{E} \max\{Y_1, \dots, Y_m\}, \quad Y \sim N(0, \Sigma). \quad (1.4)$$

Thus, the *asymptotic Rademacher complexity* (1.4) coincides with the expected value of *largest order statistics* of an m -dimensional normal random variable Y with zero mean and the covariance matrix

$$\Sigma_{kl} = (\mathbb{E}[f_k(Z)f_l(Z)])_{k,l=1}^m.$$

We will see that $\mathfrak{R}^a(\mathcal{F})$ admits a representation similar to (1.4) in the framework of Peng's sublinear expectation theory [9]. The characterization of $\mathfrak{R}^a(\mathcal{F})$ in terms of the viscosity solution of a G -heat equation is given in Theorem 2.2. This result is translated to the language of the sublinear expectation theory in Remark 2.4. In Section 3 we obtain the upper bound (1.2), as well as a lower bound for $\mathfrak{R}^a(\mathcal{F})$, combining viscosity solutions techniques with known estimates of the expected maximum of a Gaussian process.

2. THE MAIN RESULT

Our argumentation is based on a central limit theorem under model uncertainty (see [13]) which we now recall. Let $(\xi_i)_{i=1}^\infty$ be a sequence of d -dimensional random variables with zero mean and identity covariance matrix:

$$\mathbb{E}\xi_i = 0, \quad \mathbb{E}(\xi_i^k \xi_i^l)_{k,l=1}^d = I.$$

Let \mathfrak{A}_1^n be the set of sequences $A_1^n = (A_i)_{i=1}^n$, where A_i is a $\sigma(\xi_1, \dots, \xi_{i-1})$ -measurable random element with values in a compact set Λ of $d \times d$ matrices (A_1 is simply an element of Λ). For a bounded continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$ put

$$\mathcal{L} = \lim_{n \rightarrow \infty} \sup_{A_1^n \in \mathfrak{A}_1^n} \mathbb{E} f \left(\sum_{t=1}^n \frac{A_t \xi_t}{\sqrt{n}} \right).$$

Furthermore, let $G(S) = \frac{1}{2} \sup_{A \in \Lambda} \text{Tr}(AA^T S)$, where S belongs to the set \mathbb{S}^m of symmetric $d \times d$ matrices. Consider the G -heat equation

$$-v_t(t, x) - G(v_{xx}(t, x)) = 0, \quad (t, x) \in Q^\circ = [0, 1) \times \mathbb{R}^d, \quad (2.1)$$

with the terminal condition

$$v(1, x) = f(x). \quad (2.2)$$

By $v_{xx} = (v_{x_i x_j})_{i,j=1}^n$ we denote the Hessian matrix.

Recall that an upper semicontinuous (usc) (resp., a lower semicontinuous (lsc)) function $u : Q \mapsto \mathbb{R}$, $Q = [0, 1] \times \mathbb{R}^d$ is called a *viscosity subsolution* (resp., *supersolution*) of the problem (2.1), (2.2) if

$$u(1, x) \leq f(x), \quad (\text{resp.}, u(1, x) \geq f(x)),$$

and for any $(\bar{t}, \bar{x}) \in Q^\circ = [0, 1) \times \mathbb{R}^d$ and any test function $\varphi \in C^2(\mathbb{R}^{m+1})$ such that (\bar{t}, \bar{x}) is a local maximum (resp., minimum) point of $u - \varphi$ on Q° , the inequality

$$(-\varphi_t - G(\varphi_{xx}))(\bar{t}, \bar{x}) \leq 0 \quad (\text{resp.}, \geq 0)$$

holds true. A *continuous* function $u : Q \mapsto \mathbb{R}$ is called a *viscosity solution* of (2.1), (2.2) if it is both viscosity sub- and supersolution. The classical reference is [2].

Theorem 2.1. *Let $v : [0, 1] \times \mathbb{R}^d \mapsto \mathbb{R}$ be the unique bounded viscosity solution of (2.1), (2.2). Then $\mathcal{L} = v(0, 0)$.*

We refer to [13] for the proof of this result and the discussion of its relation to Peng's central limit theorem: [7].

For $\mathcal{F} = \{f_1, \dots, f_m\}$ let us rewrite the expression (1.1) as follows:

$$\mathfrak{R}_n(\mathcal{F}) = \sup_{z_1^n} \mathbb{E} g \left(\sum_{t=1}^n \frac{\varepsilon_t F(z_t(\varepsilon_1^{t-1}))}{\sqrt{n}} \right), \quad g(x) = \max\{x_1, \dots, x_m\}, \quad (2.3)$$

where $F = (f_1, \dots, f_m)$. Denote by Γ the closure of the set $\{F(z) : z \in \mathcal{Z}\} \subset \mathbb{R}^m$. The expression (2.3) can be represented in the form

$$\mathfrak{R}_n(\mathcal{F}) = \sup_{\gamma_1^n} \mathbb{E} g \left(\sum_{t=1}^n \frac{\varepsilon_t \gamma_t}{\sqrt{n}} \right), \quad (2.4)$$

where supremum is taken over all sequences γ_1^n , whose elements γ_t are measurable with respect to $\sigma(\varepsilon_1, \dots, \varepsilon_{t-1})$, and take values in Γ .

Theorem 2.2. *Let $v : [0, 1] \times \mathbb{R}^d \mapsto \mathbb{R}$ be the unique viscosity solution of the problem*

$$-v_t(t, x) - \frac{1}{2} \sup_{\gamma \in \Gamma} \sum_{i,j=1}^m \gamma^i \gamma^j v_{x_i x_j} = 0, \quad (2.5)$$

$$v(1, x) = g(x) = \max\{x_1, \dots, x_m\}, \quad x \in \mathbb{R}^m, \quad (2.6)$$

satisfying the linear growth condition: $|v(t, x)| \leq C(1 + |x|)$, where $|x|$ is the usual Euclidian norm of x . Then $\mathfrak{R}^a(\mathcal{F}) = v(0, 0)$.

Proof. In Theorem 2.1 it is not essential that matrices A_t are quadratic. So, to apply Theorem 2.1 to the expression (2.4), the only issue we need to overcome is the unboundedness of g .

The existence and uniqueness of a viscosity solution v of (2.5), (2.6), satisfying the linear growth condition, is well known from the theory of stochastic optimal control: see Theorem 5.2 and Theorem 6.1 of [19, Chapter 4]. Put $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, and denote by v_L the unique bounded viscosity solution of (2.5) satisfying the terminal condition

$$v_L(x) = g_L(x) := g(x) \vee L \wedge (-L), \quad x \in \mathbb{R}^d$$

instead of (2.6). We can apply Theorem 2.1 to (2.4) with g_L instead of g and $\gamma_t \in \Gamma$ instead of quadratic matrices $A_t \in \Lambda$. As far as the equation (2.1) corresponds to (2.5), we get

$$v_L(0, 0) = \lim_{n \rightarrow \infty} \sup_{\gamma_1^n} \mathbb{E} g_L \left(\sum_{t=1}^n \frac{\varepsilon_t \gamma_t}{\sqrt{n}} \right).$$

So, it is sufficient to prove the relations

$$\mathfrak{R}^a(\mathcal{F}) = \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{\gamma_1^n} \mathbb{E} g_L \left(\sum_{t=1}^n \frac{\varepsilon_t \gamma_t}{\sqrt{n}} \right), \quad v(0, 0) = \lim_{L \rightarrow \infty} v_L(0, 0). \quad (2.7)$$

The proof of the first equality (2.7) is similar to that of [1, Lemma A.11]. Put $X_n = n^{-1/2} \sum_{t=1}^n \varepsilon_t \gamma_t$. From the identity

$$g(x) = g_L(x) + (g(x) - L)I_{\{g(x) > L\}} + (g(x) + L)I_{\{g(x) < -L\}},$$

we get the inequalities

$$\mathbb{E} g(X_n) \leq \mathbb{E} g_L(X_n) + \mathbb{E}[(g(X_n) - L)I_{\{g(X_n) > L\}}],$$

$$\mathbb{E} g(X_n) \geq \mathbb{E} g_L(X_n) + \mathbb{E}[(g(X_n) + L)I_{\{g(X_n) < -L\}}].$$

Using the estimate $g(x) \leq |x|$, we obtain

$$\begin{aligned} \mathbb{E}[(g(X_n) - L)I_{\{g(X_n) > L\}}] &\leq \mathbb{E}[(|X_n| - L)I_{\{|X_n| > L\}}] = \mathbb{E}[(|X_n| - L)^+] \\ &= \int_0^\infty \mathbb{P}((|X_n| - L)^+ \geq u) du = \int_0^\infty \mathbb{P}(|X_n| \geq L + u) du \\ &= \int_L^\infty \mathbb{P}(|X_n| \geq u) du \leq \sum_{k=1}^m \int_L^\infty \mathbb{P}(|X_n^k| \geq u) du, \quad a^+ = \max\{a, 0\}. \end{aligned}$$

Since $(\varepsilon_t \gamma_t^k)_{t=1}^n$ is a martingale difference and $|\varepsilon_t \gamma_t^k| \leq b$, by the Azuma inequality (see, e.g., [15, Theorem 1.3.1]):

$$\mathbb{P}\left(\left|\sum_{t=1}^n \varepsilon_t \gamma_t^k\right| \geq \lambda\right) \leq 2 \exp\left(-\frac{\lambda^2}{2b^2 n}\right)$$

we get

$$\mathbb{P}(\sqrt{n}|X_n^k| \geq \sqrt{n}u) \leq 2 \exp\left(-\frac{u^2}{2b^2}\right).$$

It follows that

$$\mathbb{E}[(g(X_n) - L)I_{\{g(X_n) > L\}}] \leq r(L) := 2m \int_L^\infty \exp\left(-\frac{u^2}{2b^2}\right) du.$$

Similarly,

$$\mathbb{E}[(g(X_n) + L)I_{\{g(X_n) < -L\}}] \geq -r(L).$$

Thus,

$$\mathbb{E} g_L(X_n) - r(L) \leq \mathbb{E} g(X_n) \leq \mathbb{E} g_L(X_n) + r(L),$$

and we get the inequalities

$$\sup_{\gamma_1^n} \mathbb{E} g_L(X_n) - r(L) \leq \mathfrak{R}_n(\mathcal{F}) \leq \sup_{\gamma_1^n} \mathbb{E} g_L(X_n) + r(L),$$

which imply the first equality (2.7), since $r(L) \rightarrow 0$, $L \rightarrow \infty$,

Firthermore, put $G(X) = \frac{1}{2} \sup_{\gamma \in \Gamma} \sum_{i,j=1}^m X_{ij} \gamma^i \gamma^j$,

$$F(t, x, r, q, X) = \begin{cases} -q - G(X), & t \in [0, 1), \\ r - g(x), & t = 1, \end{cases} \quad (2.8)$$

and denote by

$$F_*(t, x, r, q, X) = \begin{cases} -q - G(X), & t \in [0, 1), \\ \min\{-q - G(X), r - g(x)\}, & t = 1 \end{cases}$$

the lsc envelope of $F : [0, 1] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^m \mapsto \mathbb{R}$. A usc function u is a viscosity solution of (2.5), (2.6) if and only if

$$F_*(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), \varphi_t(\bar{t}, \bar{x}), \varphi_{xx}(\bar{t}, \bar{x})) \leq 0 \quad (2.9)$$

for any $(\bar{t}, \bar{x}) \in Q$ and any test function $\varphi \in C^2(\mathbb{R}^{m+1})$ such that (\bar{t}, \bar{x}) is a local maximum point of $u - \varphi$ on Q . To prove this we only need to show that if u is a viscosity subsolution in the sense of the definition (2.9), then the inequality

$$(-\varphi_t - G(\varphi_{xx}))(1, \bar{x}) \leq 0$$

is impossible. Note, that $\widehat{\varphi} = \varphi + c(1 - t)$ is still a test function for u at $(1, \bar{x})$ for any $c > 0$. Thus,

$$c - \varphi_t(1, \bar{x}) - G(\varphi_{xx}(1, \bar{x})) \leq 0,$$

and we get a contradiction since c is arbitrary.

An advantage of the definition (2.9) is that it treats the equation and boundary condition simultaneously. As we have just seen, the correspondent boundary condition in the *viscosity sense*, given by (2.9) for $t = 1$ (cf. [2, §7]), is equivalent to the usual boundary condition in our case.

Viscosity supersolutions are considered in the same way. A viscosity solution v of (2.5), (2.6) may be termed as a viscosity solution of the equation

$$F(t, x, v(t, x), v_t(t, x), v_{xx}(t, x)) = 0, \quad (t, x) \in Q. \quad (2.10)$$

Denote v_+ (resp., v_-) the viscosity solution of (2.5), satisfying the terminal condition $v_+(1, x) = g^+(x)$ (resp., $v_-(1, x) = g^-(x)$), $x \in \mathbb{R}^m$, and the linear growth condition. By the comparison result of [3, Theorem 2.1] or [16, Theorem 5] it follows that

$$v_- \leq v_L \leq v_+ \quad \text{on } [0, 1] \times \mathbb{R}^d.$$

Hence, the upper and lower “relaxed limits” (see [2, §6], [5, Chapter 2])

$$\begin{aligned} \bar{v}(t, x) &= \limsup_{j \rightarrow \infty} \{v_L(s, y) : L \geq j, (s, y) \in Q, |s - t| + |y - x| \leq 1/j\}, \\ \underline{v}(t, x) &= \liminf_{j \rightarrow \infty} \{v_L(s, y) : L \geq j, (s, y) \in Q, |s - t| + |y - x| \leq 1/j\} \end{aligned}$$

are finite and satisfy the linear growth condition. Moreover, \bar{v} is usc, \underline{v} is lsc.

Denote by F_L the function of the form (2.8), where g is changed to g_L . The lower relaxed limit of the lsc envelope $(F_L)_*$ of F_L is F_* . By [5, Theorem 2.3.5] it follows that the function \bar{v} is a usc subsolution of (2.10). Similary, \underline{v} is an lsc supersolution of

(2.10) By the mentioned comparison results of [3] or [16] we have $\bar{v} \leq \underline{v}$. The opposite inequality is clear from the definition of \bar{v} , \underline{v} . It follows that the function $v = \bar{v} = \underline{v}$ coincides with the unique viscosity solution of (2.5), (2.6), and the second equality (2.7) holds true:

$$\lim_{L \rightarrow \infty} v_L(0, 0) = v(0, 0). \quad \square$$

Remark 2.3. As already mentioned, there is link between the problem (2.5), (2.6) and the stochastic control theory. Let $(W_t)_{t \geq 0}$ be a Brownian motion. Denote by \mathfrak{G} the set of stochastic processes γ adapted to the natural filtration of $(W_t)_{t \geq 0}$ and taking values in Γ . Consider the family of stochastic processes

$$X_s^{t,x,\gamma,i} = x + \int_t^s \gamma_u^i dW_u, \quad s \in [t, 1], \quad i = 1, \dots, m$$

and the related value function

$$v(t, x) = \sup \left\{ \mathbb{E} \max \{ X_1^{t,x,\gamma,1}, \dots, X_1^{t,x,\gamma,m} \} : \gamma \in \mathfrak{G} \right\}.$$

By Proposition 3.1 and Theorem 5.2 of [19, Chapter 4], v is a viscosity solution of (2.5), (2.6), satisfying the linear growth condition. In particular,

$$\mathfrak{R}^a(\mathcal{F}) = v(0, 0) = \sup \left\{ \mathbb{E} \max \left\{ \int_0^1 \gamma_u^1 dW_u, \dots, \int_0^1 \gamma_u^m dW_u \right\} : \gamma \in \mathfrak{G} \right\}.$$

Remark 2.4. Denote by $\text{conv } A$ the convex hull of a set A . Let us rewrite the equation (2.5) in the form

$$-v_t(t, x) - \frac{1}{2} \sup_{Q \in \Theta} \text{Tr}(Q v_{xx}(t, x)) = 0,$$

where $\Theta = \text{conv} \{ (\gamma^i \gamma^j)_{i,j=1}^m : \gamma \in \Gamma \} \subset \mathbb{S}^n$. In the framework of the sublinear expectation theory we have (see [9, Chapter II])

$$\mathfrak{R}^a(\mathcal{F}) = v(0, 0) = \hat{\mathbb{E}} \max \{ X_1, \dots, X_m \}, \quad (2.11)$$

where X is a multidimensional G -normal random variable: $X \sim \mathcal{N}(0, \Theta)$, and by $\hat{\mathbb{E}}$ we denote a sublinear expectation. Thus, $\mathfrak{R}^a(\mathcal{F})$ can be regarded as the *sublinear expected value of the largest order statistics* of a multidimensional G -normal random variable. Note, that the set Θ , characterizing the uncertainty structure of Y , coincides with the convex hull of covariance matrices of random vectors $\varepsilon \gamma$, $\gamma \in \Gamma = \{F(z) : z \in \mathcal{Z}\}$, where ε is a Rademacher random variable. We emphasize the similarity of this description with case of the Rademacher complexity $\mathfrak{R}^{a,iid}(\mathcal{F})$, considered in Section 1.

3. UPPER AND LOWER BOUNDS

To illustrate our approach, we derive upper and lower bounds for $\mathfrak{R}^a(\mathcal{F})$, combining simple comparison results for viscosity solutions of parabolic equations and known estimates of the expected maximum of a Gaussian process.

Theorem 3.1. *Let $\mathcal{F} = \{f_1, \dots, f_m\}$, where f_i are uniformly bounded $|f_i| \leq b$. Then*

$$\frac{1}{17}a(\mathcal{F}) \leq \frac{\mathfrak{R}^a(\mathcal{F})}{\sqrt{\ln m}} \leq \sqrt{2}b, \quad (3.1)$$

$$a(\mathcal{F}) = \sup_{\nu \in \mathcal{P}(\mathcal{G})} \inf_{i \neq j} \left(\int_{\mathcal{Z}} (f_i(z) - f_j(z))^2 \nu(dz) \right)^{1/2},$$

where $\mathcal{P}(\mathcal{G})$ is the set of probability measures on the σ -algebra of \mathcal{G} .

Proof. Along with (2.5) consider the usual heat equation

$$-u_t(t, x) - \frac{b^2}{2} \text{Tr}(u_{xx}) = 0, \quad (t, x) \in Q^\circ \quad (3.2)$$

with the terminal condition $u(1, x) = g(x)$. The function $U = e^{1-t}u$ satisfies the equation

$$-U_t + U - \frac{b^2}{2} \text{Tr}(U_{xx}) = 0 \quad (3.3)$$

and the same terminal condition. Similarly, if v is the viscosity solution of (2.5), (2.6), then the function $V = e^{1-t}v$ satisfies the equation

$$-V_t + V - \frac{1}{2} \sup_{\gamma \in \Gamma} \sum_{i,j=1}^m \gamma^i \gamma^j V_{x_i x_j} = 0, \quad (3.4)$$

in Q° in the viscosity sense, and $V(1, x) = g(x)$.

Assume that there exists a point $(\bar{t}, \bar{x}) \in Q$ such that $(V - U)(\bar{t}, \bar{x}) > 0$. In view of the terminal conditions, we have $\bar{t} < 1$. Since U, V satisfy the linear growth condition, the function

$$(V - U)(t, x) - \frac{\varepsilon}{2}|x|^2$$

attains its maximum on Q at some point $(t_\varepsilon, x_\varepsilon)$. For ε small enough one may assume that $t_\varepsilon < 1$ by virtue of the inequality

$$\sup_{(t,x) \in Q} \left((V - U)(t, x) - \frac{\varepsilon}{2}|x|^2 \right) \geq (V - U)(\bar{t}, \bar{x}) - \frac{\varepsilon}{2}|\bar{x}|^2 > 0. \quad (3.5)$$

By the definition, $U + \varepsilon|x|^2/2$ is a test function for the viscosity subsolution V of (3.4) at $(t_\varepsilon, x_\varepsilon)$. Hence,

$$\left(-U_t + V - \frac{1}{2} \sup_{\gamma \in \Gamma} \langle (U_{xx} + \varepsilon I) \gamma, \gamma \rangle \right) (t_\varepsilon, x_\varepsilon) \leq 0, \quad (3.6)$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^m . From an explicit representation of u :

$$u(t, x) = \frac{1}{(b\sqrt{2\pi(1-t)})^m} \int_{\mathbb{R}^m} \exp\left(-\frac{|y|^2}{2b^2(1-t)}\right) g(x+y) dy$$

and the convexity of g it follows that U is convex in x . Thus, U_{xx} is non-negative definite and

$$\sup_{\gamma \in \Gamma} \langle U_{xx}(t_\varepsilon, x_\varepsilon) \gamma, \gamma \rangle \leq \sup_{|\gamma| \leq b} \langle U_{xx}(t_\varepsilon, x_\varepsilon) \gamma, \gamma \rangle \leq b^2 (\text{Tr } U_{xx})(t_\varepsilon, x_\varepsilon). \quad (3.7)$$

From the inequalities (3.6), (3.7) and the equation (3.3), we get

$$V(t_\varepsilon, x_\varepsilon) \leq \left(U_t + \frac{b^2}{2} (\text{Tr } U_{xx}) \right) (t_\varepsilon, x_\varepsilon) + \frac{b^2}{2} \varepsilon = U(t_\varepsilon, x_\varepsilon) + \frac{b^2}{2} \varepsilon.$$

Combining this with (3.5):

$$0 < (V - U)(\bar{t}, \bar{x}) \leq \frac{\varepsilon}{2} |\bar{x}|^2 + (V - U)(t_\varepsilon, x_\varepsilon) \leq \frac{\varepsilon}{2} |\bar{x}|^2 + \frac{b^2}{2} \varepsilon,$$

we get a contradiction by letting $\varepsilon \rightarrow 0$.

Thus, $V \leq U$. In particular, $\mathfrak{R}^a(\mathcal{F}) = v(0, 0) \leq u(0, 0)$. To get the right inequality (3.1) we use the probabilistic representation of u :

$$\mathfrak{R}^a(\mathcal{F}) \leq u(0, 0) = \mathbb{E}g(bW_T) = b\mathbb{E} \max\{W_1^1, \dots, W_1^m\} \leq b\sqrt{2 \ln m},$$

where W_1^i are independent standard normal random variables. The last inequality is taken from [1] (Lemma A.13).

To obtain the left inequality (3.1) compare the representations (1.4) and (2.11). Since $\Sigma \subset \Theta$, we conclude that $\mathfrak{R}^{a, iid}(\mathcal{F}) \leq \mathfrak{R}^a(\mathcal{F})$. As in the first part of the proof, this is a consequence of a comparison result: see [8]. Applying to (1.4) the Sudakov inequality (see [6, Lemma 5.5.6], [17, Lemma 2.1.2]), and taking into account that Z is arbitrary, we get

$$\mathfrak{R}^a(\mathcal{F}) \geq \frac{1}{17} a(\mathcal{F}) \sqrt{\ln m},$$

$$\begin{aligned} a(\mathcal{F}) &= \sup_Z \inf_{i \neq j} (\mathbb{E}(Y_i - Y_j)^2)^{1/2} = \sup_Z \inf_{i \neq j} (\mathbb{E}(f_i(Z) - f_j(Z))^2)^{1/2} = \\ &= \sup_{\nu \in \mathcal{P}(\mathcal{Z})} \inf_{i \neq j} \left(\int_{\mathcal{Z}} (f_i(z) - f_j(z))^2 \nu(dz) \right)^{1/2}. \end{aligned} \quad \square$$

Assuming that $a(\mathcal{F}) \geq c > 0$ uniformly in m , from (3.1) we see that $\mathcal{R}(\mathcal{F}) \sim \sqrt{\ln m}$ for large cardinality of \mathcal{F} . The factor $1/17$ in the lower bound (3.1) can be refined: see [4, Section 2.3].

It would be interesting to extend the representation (2.11) to the case of an infinite function class \mathcal{F} .

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